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DEVELOPMENT OF METHODS FOR SOLVING THE TASKS OF THE CONTINIUM LINEAR PROGRAMMING USING LEGENDRE POLYNOMIALS

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Abstract
The article demonstrates the theoretical foundations of the mathematical apparatus − the continuum of linear programming. It demonstrates a technique for solving problems with the use of orthogonal systems of functions. The article was an exact solution of the problem of variational calculus to linear constraints. The purpose of the work is to develop accurate methods of solving the problem in the class of Legendre polynomials. The study demonstrates an ability to build the exact solution of the problem and the conditions under which the decision is allowed. Based on the properties of Legendre polynomials, an exact solution of the problem of continual linear programming is provided, in which the integrands and functional limitations are presented in rows of finite degree. Analytically, it is proven that the solution obtained is a limiting case of the linear combination of delta functions. It is shown that the parameters of the optimization problem of finding the unknown functions plan contains half of the variables than in the canonical method. Recommendations are given for the construction of the optimization algorithm. There is a possibility of extending the proposed technology solution in the direction of using other systems of orthogonal polynomials.
Keywords: continual linear programming; Legendre polynomials; delta function.

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РАЗРАБОТКА МЕТОДОВ РЕШЕНИЯ ЗАДАЧ КОНТИНУАЛЬНОГО ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ С ИСПОЛЬЗОВАНИЕМ ПОЛИНОМОВ ЛЕЖАНДРА

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Аннотация
В статье представлены теоретические основы математического аппарата континуум линейного программирования. Показанный метод решения задач континуального линейного программирования с использованием ортогональных систем функций. В статье получено точное решением задачи вариационного исчисления при наличии линейных ограничений. Целью работы является разработка точных методов решения задачи в классе полиномов Лежандра. Продемонстрирована возможность построить точное решение задачи и условия, при которых решение существует. Аналитически доказано, что полученное решение следует искать в виде линейной комбинации дельта-функций. Даны рекомендации по построению алгоритма оптимизации. Указано на возможность использовать при решении данных задач других систем ортогональных многочленов.
Ключевые слова: континуальное линейное программирование; многочлены Лежандра; дельта-функция.

Introduction

The simplest of these tasks can be solved by conventional means, if it is the class of functions in which a decision is based [1, p.25]. It was found that the quality of the solution is higher the more "a delta" has a character of its analytical solution. It is shown that the exact solution of the simplest problem of continual linear programming must be sought in the class of delta functions [1, p.48]. In general, the synthesis problem can be formulated as follows: Let the input of a linear system with a known frequency response signal is, the spectral power is described by the function. Find a function that maximizes functional:

$$\int_{R_1}^{R_2} C(R)X(R)dR, \qquad R \in [R_1, R_2] \quad (1)$$

and satisfying the limits for the spectral power of the signal system.

$$\int_{R_1}^{R_2} A_i(R)X(R)dR = B_i\ ,\quad i = 1,2,...,m \quad (2)$$

$$X(t) \geq 0, \quad (3)$$

where $C(R)$, $A_i(R)$ - are integrated in the R function. The research in this article is dedicated to the development of exact methods for solving the above problem of continual linear programming (1)-(4).

1. The formulation of the problem in the class of Legendre polynomias

Let's introduce the variable t [1, p.24]:

$$t = \frac{2R - R_2 - R_1}{R_2 - R_1},\ R = \frac{t(R_2 - R_1) + R_2 + R_1}{2} \quad (4)$$

with provision of which, we will get the next presentation of our problem:

$$\int_{R_1}^{R_2} C(R)X(R)dR = \frac{(R_2 - R_1)}{2}\int_{-1}^{1} C\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right)\times$$

$$\times X\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right)dt\ , \quad (5)$$

$$\int_{R_1}^{R_2} A_i(R)X(R)dR = \frac{(R_2 - R_1)}{2}\int_{-1}^{1} A_i\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right)\times$$

$$\times X\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right)dt\ , \quad (6)$$

$$X\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right) > 0\ . \quad (7)$$

Using the labeling for $c(t)$ $a_i(t)$, $x(t)$:

$$c(t) = \frac{(R_2 - R_1)}{2}\cdot C\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right),$$

$$a_i(t) = \frac{(R_2 - R_1)}{2}\cdot A_i\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right), \quad (8)$$

$$x(t) = X\left(\frac{t(R_2 - R_1) + R_2 + R_1}{2}\right) \geq 0\ . \quad (9)$$

Let's present the problem in a canonical form:

$$\int_{-1}^{1} c(t)x(t)dt \quad (10)$$

$$\int_{-1}^{1} a_i(t)x(t)dt = b_i\ ,\qquad i = 1,2,..,m\ , \quad (11)$$

$$x(t) \geq 0\ , \quad (12)$$

where c(t),a(t) because of integrability of $C(R)$, $A_i(R)$ on the interval are the integrable functions on the interval $t \in [-1;1]$.

If the limits of the integration in (1) and (2) are respectively different and equal R_{21}, R_{11} for (1) and R_{22}, R_{12} (2) then selecting, $R_2 = \max\{R_{ij}\}$, $R_1 = \min\{R_{ij}\}$, ($i = 1,2$; $j = 1,2$) and setting function $C(R) = 0$ for $R \notin [R_{11}, R_{21}]$, $A_i(R) = 0$ for $R \notin [R_{12}, R_{22}]$, when the problem is reduced to the form (1)-(3).

2. Solution of the problem in the class of Legendre polynomials

The solution $x(t)$ is presented on the interval $t \in [-1;1]$ in the form of an expansion in Legendre polynomials $P_n(t)$ [4, p.64–74]:

$$x(t) = \sum_{n=0}^{\infty}\gamma_n P_n(t),\ \gamma_n = \frac{2n+1}{2}\int_{-1}^{1} x(t)P_n(t)dt\ ,$$

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n (t^2 - 1)^n}{dt^n}\ . \quad (13)$$

Representing the specified functions $c(t), a_i(t)$ (8) in the form of a convergent series, $i = 1,2,...,m$:

$$c(t) = \sum_{n=0}^{\infty}\sigma_n P_n(t)\ ,\sigma_n = \frac{2n+1}{2}\int_{-1}^{1} c(t)P_n(t)dt\ , \quad (14)$$

$$a_i(t) = \sum_{n=0}^{\infty}\alpha_{i,n} P_n(t),\ \alpha_{i,n} = \frac{2n+1}{2}\int_{-1}^{1} a_i(t)P_n(t)dt\ , \quad (15)$$

let's comply the integration (10), (11) of the formula [4, p.71]:

$$\int_{-1}^{1} c(t)x(t)dt = \int_{-1}^{1}\sum_{n=0}^{\infty}\sigma_n P_n(t)\sum_{k=0}^{\infty}\gamma_k P_k(t)dt =$$

$$= \sum_{n=0}^{\infty}\sigma_n\gamma_n\int_{-1}^{1} P_n^{\,2}(\tau)d\tau = \sum_{n=0}^{\infty}\sigma_n\gamma_n\frac{2}{2n+1}\ , \quad (16)$$

$$\int_{-1}^{1} c(t)x(t)dt = \int_{-1}^{1}\sum_{n=0}^{\infty}\sigma_n P_n(t)\sum_{k=0}^{\infty}\gamma_k P_k(t)dt =$$

$$= \sum_{n=0}^{\infty}\sigma_n\gamma_n\int_{-1}^{1} P_n^{\,2}(\tau)d\tau = \sum_{n=0}^{\infty}\sigma_n\gamma_n\frac{2}{2n+1}\ , \quad (17)$$

and we'll get the statement of the problem, which consists in maximizing function

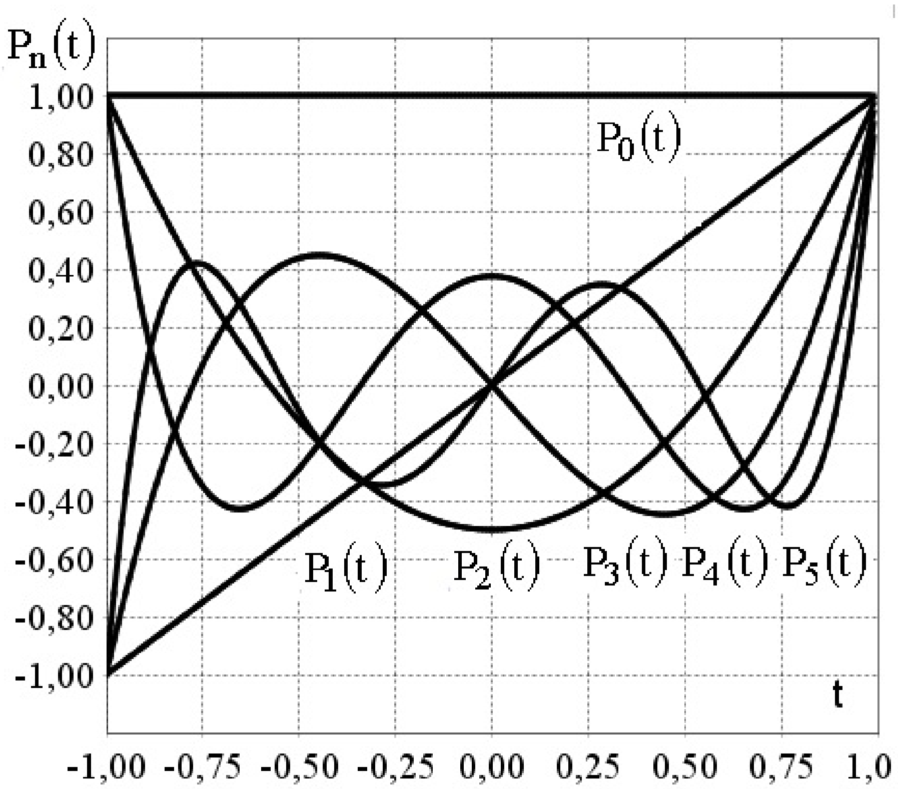

Fig. 1. Legendre's polynomials $P_n(t)$, n=0, 1, 2, 3, 4, 5

$$\sum_{n=0}^{\infty}\sigma_n\gamma_n \frac{2}{2n+1} \to \max , \qquad (18)$$

satisfying constraints

$$\sum_{n=0}^{\infty}\alpha_{i,n}\gamma_n \frac{2}{2n+1} = b_i , i = 1,2,...,m , \qquad (19)$$

$$\sum_{n=0}^{\infty}\gamma_n P_n(t) \geq 0 , \quad t \in [-1,1]. \qquad (20)$$

Let's deal with the solution of the problem (10)-(12) for the case when the functions $C(R)$, $A_i(R)$ are represented by polynomials of degree *N1*, *N2*:

$$C(R) = \sum_{n=0}^{N_1}\Pi_n R^n , \quad A_i(R) = \sum_{n=0}^{N} W_{i,n} R^n , \qquad (21)$$

with coefficients Π_n and $W_{i,n}$. These include the problem of the LPC, which limits (2) defines the moment of order $\int_{R_1}^{R_2} R^i X(R) dR = B_i$ to the range of variation of a random variable $R \in [R_1, R_2]$, [1, p.13-17]. The expression t^N can be represented by the expansion of the form [5, p.79]:

$$t^N = \sum_{n=0}^{N}\gamma_n P_n(t), \ \gamma_N = \frac{2^N (N!)^2}{(2N)!} , \ \gamma_n = 0 , \qquad (22)$$

when $(N-n)$ is odd number,

$$\gamma_n = \frac{(2n+1)\cdot 2^n N!\left(\frac{N+n}{2}\right)!}{\left(\frac{N-n}{2}\right)!\cdot(N+n+1)!} , \qquad (23)$$

when $(N-n)$ is even integer, which contains the finite number N Legendre's polynomials $P_n(t)$.

3. The usage of Legendre polynomials for constructing exact solutions

Let's imagine, that $C(R)$ and $A_i(R)$ are polynomials R degree N_1 and N_2, $N_1 > N_2$. In the view of the identities (22), (23) with (8) functions $c(t)$, $a_i(t)$ are polynomials and the degree N_1 and N_2 with the expansion coefficients σ_n, $\alpha_{i,n}$, respectively (14) and (15). We represent the right side of (11) integral with the integrand containing Legendre polynomials, $i = 1,2,...,m$:

$$b_i = \int_{-1}^{1}\sum_{n=0}^{N}\beta_{i,n}\sum_{k=0}^{\infty}\frac{2k+1}{2}P_k(t_n)P_k(t)dt , \quad (24)$$

$$\int_{-1}^{1} a_i(t)x(t)dt = \int_{-1}^{1}\sum_{n=0}^{N_2}\alpha_{i,n}P_n(t)\sum_{k=0}^{\infty}\gamma_k P_k(t)dt =$$

$$= \int_{-1}^{1} \sum_{n=0}^{N} \beta_{i,n} \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) dt\,, \quad (25)$$

and we will get an equation that must be solved for γ_k, $i = 1,2,...,m$:

$$\int_{-1}^{1} \sum_{n=0}^{N_2} \alpha_{i,n} P_n(t) \sum_{k=0}^{\infty} \gamma_k P_k(t) dt =$$

$$= \int_{-1}^{1} \sum_{n=0}^{N} \beta_{i,n} \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) dt\,. \quad (26)$$

In the view of orthogonal polynomials $P_k(t)$ expression (26) can be presented as:

$$\int_{-1}^{1} \sum_{k=0}^{\infty} \alpha_{i,k} P_k(t) \gamma_k P_k(t) dt =$$

$$= \int_{-1}^{1} \sum_{n=0}^{N} \beta_{i,n} \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) dt\,,\ \alpha_{i,k} = 0\,, \quad (27)$$

when $k > N_2$. The solving for γ_k we will present as next

$$\gamma_k = \sum_{n=0}^{N} \omega_n \frac{2k+1}{2} P_k(t_n). \quad (28)$$

By substituting (28) into (27) we obtain

$$\int_{-1}^{1} \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} \frac{2k+1}{2} P_k(t_n) P_k(t) P_k(t) dt =$$

$$= \int_{-1}^{1} \sum_{n=0}^{N} \beta_{i,n} \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) dt\,. \quad (29)$$

The solving is next:

$$x(t) = \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t), \quad (30)$$

where ω_n the constants of integration. The right side of formula (30) can be written as follows:

$$\sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) = \delta(t - t_n), \quad (31)$$

that allows to present the solution (30) this way:

$$x(t) = \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(t_n) P_k(t) = \sum_{n=0}^{\infty} \gamma_k P_k(t) =$$

$$\sum_{n=0}^{N} \omega_n \delta(t - t_n) \quad (32)$$

Indeed, integrating with the left side of (29), taking into account (15) we obtain:

$$\int_{-1}^{1} \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} \frac{2k+1}{2} P_k(t_n) P_k(t) P_k(t) dt =$$

$$= \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} \frac{2k+1}{2} P_k(t_n) \int_{-1}^{1} P_k(t) P_k(t) dt =$$

$$= \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} P_k(t_n) = \sum_{n=0}^{N} \omega_n a_i(t_n). \quad (33)$$

On the other hand, using (32), a similar result can be written:

$$\int_{-1}^{1} \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} \frac{2k+1}{2} P_k(t_n) P_k(t) P_k(t) dt =$$

$$\int_{-1}^{1} \sum_{n=0}^{N} \omega_n \sum_{k=0}^{\infty} \alpha_{i,k} P_k(t) \delta(t - t_n) dt =$$

$$= \int_{-1}^{1} \sum_{n=0}^{N} \omega_n \delta(t - t_n) a_i(t) dt = \sum_{n=0}^{N} \omega_n a_i(t_n). \quad (34)$$

An important fact is that if the coefficients γ_n of one and quite simply determined by (11), (25), $i = 1,2,...,m$:

$$\int_{-1}^{1} a_i(t) x(t) dt = \sum_{n=0}^{N_2} \alpha_{i,n} \gamma_n \frac{2}{2n+1} = b_i\,. \quad (35)$$

The rest $(N_1 - N_2)$ of the coefficients γ_n are determined from (18) with the unambiguous representation (28), (31). It should be noted that by virtue of (28) to determine γ_n the condition $x(t) \ge 0$ will be satisfied when $N \to \infty$. This condition limits the range of acceptable solutions (10)-(12).

4. An algorithm for solving the simplest problem

Consider the exact solution of the problem (1)-(3) for functions $C(R) = 1 - R^2$, $A_0(R) = 1$ [1, p.24]:

$$\int_{-1}^{1} (1 - R^2) X(R) dR\,, \int_{-1}^{1} X(R) dR = S_0\,, X(t) \ge 0\,,$$

$$R \in [-1;1]. \quad (36)$$

Using (8) and (9) represent the task (36) in the form:

$$\int_{-1}^{1} c(t) x(t) dt\,,\ c(t) = 1 - t^2\,, \quad (37)$$

$$\int_{-1}^{1} a_0(t) x(t) dt = S_0\,,\ a_0(t) = 1\,, \quad (38)$$

$$x(t) \ge 0,\ R \in [-1;1] \quad (39)$$

Functions $c(t)$ (37), $a_0(t)$ (38) in a series in Legendre polynomials

$$c(t) = P_0(t) - \left(\frac{1}{3} P_0(t) + \frac{2}{3} P_2(t) \right) =$$

$$= \frac{2}{3} P_0(t) - \frac{2}{3} P_2(t),\ a_0(t) = P_0(t), \quad (40)$$

where $1 = P_0(t)$, $t^2 = \left(\frac{1}{3}P_0(t) + \frac{2}{3}P_2(t)\right)$.

This allows you to write the problem (37)-(39) as (18)-(20) as follows:

$$\frac{4}{3}\gamma_0 - \frac{4}{15}\gamma_2 \to \max, \qquad (41)$$

$$2\gamma_0 = S_0 \Rightarrow \gamma_0 = \frac{S_0}{2}, \qquad (42)$$

$$\sum_{n=0}^{\infty}\gamma_n P_n(t) \geq 0,\ t \in [-1;1]. \qquad (43)$$

Since (42) $a_0(t)$ is represented by one member of the decomposition $a_0(t) = P_0(t)$ (40), the search γ_2 for maximizing the functional (37) use the equation (32), $N = 1$, equating the coefficients in different k:

$$\gamma_0 = \frac{\omega_0}{2},\ k = 0,\ \omega_0 = 2\gamma_0 = S_0, \qquad (44)$$

$$\gamma_2 = \omega_0 \frac{5}{2}P_2(t_n),\ k = 2. \qquad (45)$$

Substituting γ_0 (42), γ_2 (43) to (41), we define t_n in which the expression (41) takes the maximum value:

$$\frac{4}{3}\gamma_0 - \frac{4}{15}\omega_0\frac{5}{2}P_2(t_n) = \frac{4}{3}\gamma_0 - \frac{2}{3}\omega_0 P_2(t_n) = = \frac{4}{3}\gamma_0(1 - P_2(t_n)) \to \max \qquad (46)$$

The function has a maximum value of the functional (37) is reached at $P_2(t_n) \to \min$. Where should be $P_2(t_n = 0) = -0.5$. Thus:

$$M = \int_{-1}^{1} c(t)x(t)dt = \frac{4}{3}\gamma_0 - \frac{4}{15}\gamma_2 \to \max = S_0, \qquad (47)$$

when

$$x(t) = S_0\sum_{k=0}^{\infty}\frac{2k+1}{2}P_k(t_n)P_k(t). \qquad (48)$$

Figure 2 shows the behavior of the functional value of M (47) depending on the number of polynomials in the solution $x(t)$ (48) for $S_0 = 1$. It can be seen that when $k > 2$ no functional change its value remains constant. This result is due to the finite number of polynomials $P_n(t)$ in the expansion of $c(t)$ (40).

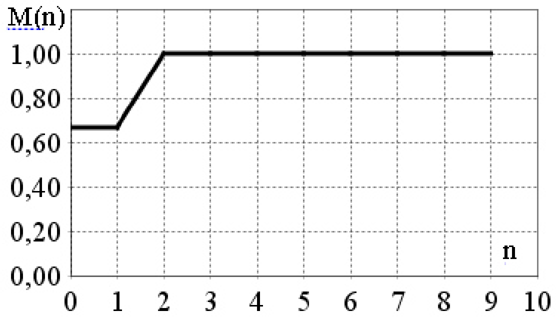

Fig. 2. The dependence of the value of the functional *M*(*n*) (46) on the number n of the Legendre polynomials $P_n(t)$ in the solution $x(t)$ (47) for the case $S_0 = 1$

The solution (37)-(39) is shown in Figure 3. It can be seen that an increase in terms of expansion solution presented in the form of (32) takes a delta character. Each of the solutions presented provides a value of the functional (10) *M*=1 if the constraint for $S_0 = 1$ (11). However, compliance with the inequality $x(t) \geq 0$ is satisfied for $N \to \infty$.

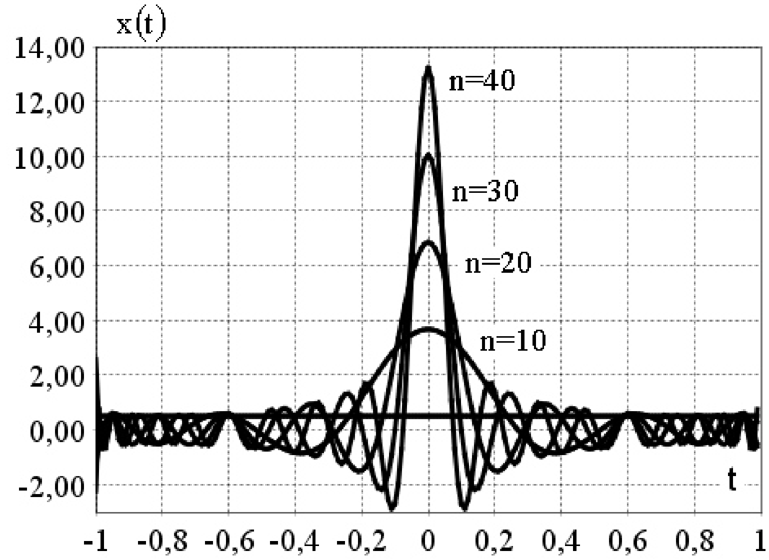

Fig. 3. Accuracy representation $x(t)$ (47) for the case $S_0 = 1$ according to the n in the expansion of Legendre polynomials $P_n(t)$

Conclusion

Investigation of properties of solutions of the continuum of linear programming have led to the following conclusions, which are as follows:

1. If the functions $c(t)$ and $a_i(t)$ are represented by finite power series of the form (21), the problem (10)-(12) has an exact solution $x(t) = \sum_{n=0}^{\infty} \gamma_n P_n(t)$ presented Legendre polynomials (13).

2. For calculating the coefficients γ_k determining the form of the solution $x(t)$, we get easy and convenient for practical calculation formula (28).

3. Analitically shown that the solution $x(t) = \sum_{n=0}^{\infty} \gamma_n P_n(t)$ degenerates into a superposition of delta functions, which confirms the conclusions drawn in [1, p.126] – the solution (10)-(12) must be sought as a linear combination of delta functions.

4. Usage of the orthogonal system of functions allowed to formulate noniteration ability to exact solution (30) of the problem (10)-(12).

5. If for $c(t)$ (14) and $a_i(t)$ (15) do not permit expansion in an orthogonal set of functions in a finite series, then the solution may be found with a given accuracy. Its value is determined by the accuracy of representation of the specified functions $c(t)$ and $a_i(t)$ finite power series (14), (15).

References

1. Raskin L.G., Kirichenko I.O., Seraja O.V. Applied Linear Continual Programming. Kharkiv: Oberig, 2014. 292 p.
2. Lesnik C., Opilski Z., Pustelny T. The Numerical Synthesis of a Radar Signal Based on Iterative Method. Acta Physica Polonica. 2011. №4 (120). Pp. 693-697.
3. Kabakchiev A., Kyovtorov V., Bedzev B., Numerical Analysis of Different Communication Signals. Cybernetics and Information Technologies. 2010. №4 (10). C. 75-90.
4. Dancig D.V. Linear Programming, its Application and Generalisation. Moscow: PROGRESS, 1966. 600 p.
5. Lebedev N.N. Special Functions and their Applications. Moscow-Leningrad: Fizmatgiz, 1963. 360 p.